Stochastic Processes and their Applications 21 (1986) 327-338
North-Holland

A LIMIT THEOREM FOR ALMOST MONOTONE SEQUENCES OF RANDOM VARIABLES

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Received 2 May 1985
Revised 8 October 1985

In this paper we consider families $(X_{m,n})$ of random variables which satisfy a subadditivity condition of the form $X_{0,n+m} \leqslant X_{0,n} + X_{n,n+m} + Y_{n,n+m}$, $m, n \geqslant 1$. The main purpose of this paper is to give conditions which are sufficient for the a.e. convergence of $((1/n)X_{0,n})$. Restricting ourselves to the case when $(X_{0,n})$ has certain monotonicity properties, we derive the desired a.e. convergence of $((1/n)X_{0,n})$ under moment hypotheses concerning $(Y_{m,n})$ which are considerably weaker than those in Derriennic [4] and Liggett [15] (in [4, 15] no monotonicity assumptions were imposed on $(X_{0,n})$). In particular, it turns out that the sequence $(E[Y_{0,n}])$ may be allowed to grow almost linearly. We also indicate how the obtained convergence results apply to sequences of random sets which have a certain subadditivity property.

L^1-convergence * a.e. convergence * subadditive ergodic theorem * almost subadditive sequence * superstationary sequence * percolation * entropy * random sets

1. Introduction

Throughout this paper we consider families $X = (X_{m,n}) \subset L^1$ and $Y = (Y_{m,n}) \subset L^1_+$ of real random variables which are defined on some probability space $(\Omega, \mathcal{A}, P)$ (the index set of families like $(X_{m,n})$ will always be $I = \{(m, n) \mid 0 \leqslant m < n,\ m, n$ integers$\}$, and we often write X_n and Y_n instead of $X_{0,n}$ and $Y_{0,n}$, respectively). Our aim is to derive conditions in terms of X and Y which ensure that $((1/n)X_n)$ converges a.e. or in L^1. Kingman [9, 10] has shown that if the *subadditivity condition*

$$X_{k,n} \leqslant X_{k,m} + X_{m,n}, \quad (k, m), (m, n) \in I \tag{1.1}$$

holds, $((1/n)X_n)$ converges a.e. and in L^1 provided X is stationary in some sense and satisfies a certain moment condition. Recently, Kingman's subadditive ergodic theorem has been generalized by Derriennic [4] who derived a corresponding result for a certain class of processes X which are *almost subadditive* (w.r.t. Y), i.e. satisfy

$$X_{k,n} \leqslant X_{k,m} + X_{m,n} + Y_{m,n}, \quad (k, m), (m, n) \in I. \tag{1.2}$$

Derriennic's result (extending an L^1-convergence result of [18]) might be considered a stochastic analogue of the following simple result on real sequences, which will be used later.

Lemma 1.1. *Let $(a_n) \subset R$ and $(c_n) \subset R_+$ be sequences satisfying the following two conditions*:

$$a_{m+n} \leq a_m + a_n + c_n, \quad m, n \geq 1, \quad \textit{and} \quad \lim_n \frac{1}{n} c_n = 0.$$

Then $\lim_n (1/n) a_n = a$ *exists and satisfies* $-\infty \leq a < \infty$.

For the simple proof cf. [4]. Related results can be found in [7, 8].

Derriennic's proof of the a.e. convergence of $((1/n)X_n)$ essentially follows Kingman's arguments in [9] and requires that Y satisfies the moment condition

$$\sup_n E[Y_n] < \infty.$$

It has been shown in [20] that this condition can be relaxed to

$$\liminf_n \frac{1}{n} \sum_{i=1}^{n} E[Y_i] < \infty. \tag{1.3}$$

One might ask whether (1.3) can be relaxed further by imposing additional restrictions on X. We will show (see Theorem 3.3 in Section 3 which is our main result) that provided (X_n) has certain monotonicity properties (to be introduced in Section 3) one can obtain an a.e. convergence result for $((1/n)X_n)$ where (1.3) may be replaced e.g. by the requirement that

$$E[Y_n] = \mathrm{O}\left(\frac{n}{\log n\,(\log\log n)^{1+\delta}}\right) \quad \text{as } n \to \infty \tag{1.4}$$

holds for some constant $\delta > 0$.

The proof of Theorem 3.3 is based on a construction in [5, 6] which recently has been used by Liggett [15, 16] to give a fairly simple proof of a new version of Kingman's result in which the subadditivity and stationarity assumptions are relaxed. In fact, in [15] it is required that instead of (1.2) the condition

$$X_{0,n+m} \leq X_{0,n} + X_{n,n+m}, \quad m, n \geq 1,$$

holds which turns out *not* to be equivalent to (1.2) (in the case $Y_{m,n} \equiv 0$, $(m, n) \in I$) under the weaker stationarity assumptions in [15] (an example is provided in [15]; see also (1.5) below). Theorem 3.3 in Section 3 shows that provided (X_n) has certain monotonicity properties mentioned above (in [4, 15] no monotonicity assumptions at all were imposed on (X_n)) the a.e. convergence of $((1/n)X_{0,n})$ still holds under certain stationarity and moment hypotheses weaker than those in [4, 15]. In particular, instead of (1.2) it is required in Theorem 3.3 that

$$X_{0,n+m} \leq X_{0,n} + X_{n,n+m} + Y_{n,n+m}, \quad m, n \geq 1,$$

holds where (Y_n) is supposed to satisfy a condition like (1.4). Katznelson and Weiss [13] have, perhaps, given the most elementary proof of Kingman's subadditive ergodic theorem to date but apparently their method is not suitable for obtaining our Theorem 3.3.

In Section 2 we derive a general L^1-convergence result for $((1/n)X_n)$ extending Theorem 1 in [4], which is interesting in its own right. In order to prove Theorem 3.3, we also need a somewhat surprising convergence result for real sequences (Theorem 3.2 of Section 3) its proof being deferred to Section 4.

At the end of Section 3 we indicate how our main result (Theorem 3.3) can be applied to obtain a very general a.e. convergence result for certain families of random sets which are almost subadditive and have certain monotonicity properties.

Finally we would like to point out that Derriennic [4] has used his almost subadditive theorem in order to give a new proof of the Shannon–McMillan–Breiman theorem of information theory (here, the process $(X_{m,n})$ involved has the property that (X_n) is increasing). On the other hand, Liggett [15] has applied his convergence result to a certain process $(X_{m,n})$ arising in oriented percolation [6] (not satisfying all of Kingman's [9] hypotheses). Here, X_n, $n \geq 1$, is given by

$$X_n = \max\{m \mid \text{there is an active path from } (k, 0) \text{ to } (m, n) \text{ for some } k \geq 0\}. \tag{1.5}$$

It is easily seen that

$$X_{n+1} \leq X_n + 1, \quad n \geq 1.$$

Hence in our terminology (cf. Section 3) (X_n) is quasi decreasing (it was, in fact, this example which suggested the introduction of our notion of quasi monotonicity), and our main result (Theorem 3.3) applies to $(X_{m,n})$ provided the probability of an edge to be open is sufficiently close to one (see [15] for further details).

2. L^1-convergence of $((1/n)X_n)$

First we formulate an L^1-convergence result for $((1/n)X_n)$ (extending Theorem 1 of [4]). In this section (X_n) is not supposed to have any monotonicity properties. A sequence $Z_1, Z_2, \ldots$ of (real) random variables defined on a common probability space is called *superstationary* [14] if the sequence $Z_2, Z_3, \ldots$ is *stochastically smaller* than the sequence $Z_1, Z_2, \ldots$ which means (cf. [11, 12]) that for all functions $f: R^\infty \to R$ which are bounded, measurable and increasing we have

$$E[f(Z_2, Z_3, \ldots)] \leq E[f(Z_1, Z_2, \ldots)].$$

We will put $a^+ = \max(a, 0)$, $a \in R$; $\|\cdot\|_1$ will denote the L^1-norm. It will be convenient to define $X_0 \equiv 0$, $X_{n,n} \equiv 0$, $Y_{n,n} \equiv 0$, $n \geq 0$.

Remark. Note that in the above definition of the relation 'stochastically smaller', 'measurable' can equivalently be replaced by 'continuous'.

Theorem 2.1. *Let K and c_n, $n \geq 0$, be positive constants. Assume that the following* *conditions hold*:

$$E[(X_{n+m}-X_n-X_{n,n+m})^+]\leqslant c_m,\quad m,n\geqslant 0. \tag{2.1}$$

$$\lim_n \frac{1}{n}c_n=0. \tag{2.2}$$

$$E[X_{jn,(j+1)n}]\geqslant -Kn,\quad j\geqslant 0,\ n\geqslant 1. \tag{2.3}$$

$$E[X_n]\geqslant E[X_{m,n+m}],\quad m,n\geqslant 1. \tag{2.4}$$

$$E[X_n^+]\geqslant E[X_{m,n+m}^+],\quad m,n\geqslant 1. \tag{2.5}$$

Finally assume that for each $n\geqslant 1$ *the sequence* $X_{0,n}, X_{n,2n}, X_{2n,3n},\dots$ *is superstationary. Then* $((1/n)X_n)$ *converges in* L^1 *to the random variable*

$$\bar{X}_\infty=\lim_n \frac{1}{n}\left(\lim_k \frac{1}{k}\sum_{j=0}^{k-1} X_{jn,(j+1)n}\right), \tag{2.6}$$

the inner limit existing a.e. and in L^1, *and the outer limit existing in* L^1.

Remark. Note that (2.4), (2.5) as well as the final assumption in Theorem 2.1 are satisfied if X is assumed to be superstationary [11, 12]. Hence, Theorem 2.1 generalizes the L^1-part of Abid's [1] convergence result.

Proof. Let us put

$$\gamma_n=E[X_n],\quad n\geqslant 1. \tag{2.7}$$

We have by (2.4) and (2.1)

$$\gamma_{n+m}\leqslant \gamma_n+\gamma_m+c_m,\quad n,m\geqslant 1.$$

Hence it follows from (2.2), (2.3) and Lemma 1.1 that

$$\gamma=\lim_n \frac{1}{n}\gamma_n \tag{2.8}$$

exists and is finite (γ is sometimes called the *time constant* of X). Now fix any $n\geqslant 1$, and write $m\geqslant 1$ in the form

$$m=kn+r,\quad 0\leqslant r<n \tag{2.9}$$

(r being an integer). Proceeding as in the proof of Theorem 1 in [4] we get

$$E\left[\left(X_m-\sum_{j=0}^{k-1} X_{jn,(j+1)n}\right)^+\right]\leqslant (k-1)c_n+c_r+\|X_r\|_1,\quad m\geqslant 1. \tag{2.10}$$

Since the sequence $X_{0,n}, X_{n,2n}, X_{2n,3n},\dots$ satisfies the hypotheses of Krengel's [14] ergodic theorem,

$$\lim_k \frac{1}{k}\sum_{j=0}^{k-1} X_{jn,(j+1)n}=\bar{X}_n\quad \text{exists a.e. and in } L^1. \tag{2.11}$$

Using (2.10), (2.11) as well as (2.3) and (2.5), we arrive at

$$\limsup_m E\left[\left(\frac{1}{m}X_m - \frac{1}{n}\bar{X}_n\right)^+\right] \leqslant \frac{1}{n}c_n.$$

The proof can now be finished as the proof of Theorem 1 in [4].

The next result extends Theorem 2 of [4] and will be used later (concerning the families X and Y, cf. Section 1). We denote by $F_{m,n}$, $G_{m,n}$ the distribution functions of $X^+_{m,n}$ and $Y_{m,n}$, respectively.

Theorem 2.2. *Let Conditions* (2.3) *and* (2.4) *of Theorem* 2.1 *be satisfied. Assume that the following conditions hold:*

$$X_{n+m} \leqslant X_n + X_{n,n+m} + Y_{n,n+m}, \quad m, n \geqslant 1. \tag{2.12}$$

$$\lim_n \frac{1}{n} E[Y_n] = 0. \tag{2.13}$$

$$F_{0,n} \leqslant F_{m,n+m}, \quad m, n \geqslant 1. \tag{2.14}$$

$$G_{0,n} \leqslant G_{m,n+m}, \quad m, n \geqslant 1. \tag{2.15}$$

Finally assume that for each $n \geqslant 1$ *the sequences* $X_{0,n}, X_{n,2n}, X_{2n,3n}, \ldots$ *and* $Y_{0,n}, Y_{n,2n}, Y_{2n,3n}, \ldots$ *are superstationary. Then* $((1/n)X_n)$ *converges in* L^1 *to* $\bar{X}_\infty$ *given by* (2.6). *Furthermore we have*

$$\limsup_n \frac{1}{n} X_n = \bar{X}_\infty \quad a.e. \tag{2.16}$$

and

$$E\left[\limsup_n \frac{1}{n} X_n\right] = \gamma, \tag{2.17}$$

γ *denoting the time constant of* X *(given by* (2.8)).

Remark. Note that (2.14), (2.15) as well as the final assumptions of Theorem 2.2 are satisfied if X and Y are superstationary [1, 11, 12].

Proof. L^1-convergence of $((1/n)X_n)$ to $\bar{X}_\infty$ is obvious from Theorem 2.1. Hence, in order to prove (2.16) it clearly suffices to show

$$E\left[\limsup \frac{1}{n} X_n\right] \leqslant E[\bar{X}_\infty]. \tag{2.18}$$

Let $n \geqslant 1$ be fixed. By Krengel's [14] ergodic theorem

$$\lim_k \frac{1}{k} \sum_{j=0}^{k-1} Y_{jn,(j+1)n} = \bar{Y}_n \quad \text{exists a.e. and in } L^1, \tag{2.19}$$

and, by (2.15),

$$E[\bar{Y}_n] \leqslant E[Y_n]. \tag{2.20}$$

Applying the Borel–Cantelli lemma as well as (2.14) and (2.15), we get for each $s \geqslant 1$

$$\lim_k \frac{1}{k} X^+_{kn,kn+s} = 0 \quad \text{a.e.} \tag{2.21}$$

and

$$\lim_k \frac{1}{k} Y_{kn,kn+s} = 0 \quad \text{a.e.} \tag{2.22}$$

Using (2.9) and applying repeatedly (2.12), we get for $m \geqslant 1$

$$X_m \leqslant \sum_{j=0}^{k-1} (X_{jn,(j+1)n} + Y_{jn,(j+1)n}) + X^+_{kn,m} + Y_{kn,m}.$$

This, together with (2.11), (2.19), (2.21), (2.22), (2.20), (2.13) and (2.6) yields (2.18). Finally, (2.17) is clear from (2.16) since $((1/n)X_n)$ converges in L^1 to $\bar{X}_\infty$.

3. Almost everywhere convergence of $((1/n)X_n)$

In view of (2.17), the next result is a first step towards establishing an almost everywhere convergence result for $((1/n)X_n)$.

Lemma 3.1. *Let K be a positive constant. Suppose that the following conditions are satisfied*:

$$X_{n+m} \leqslant X_n + X_{n,n+m} + Y_{n,n+m}, \quad m, n \geqslant 1. \tag{3.1}$$

$$\lim_n \frac{1}{n} E[Y_n] = 0. \tag{3.2}$$

$$E[X_n] \geqslant -Kn, \quad n \geqslant 1. \tag{3.3}$$

$$E[X_n] \geqslant E[X_{m,n+m}], \quad m, n \geqslant 1. \tag{3.4}$$

$$E[X^+_1] \geqslant E[X^+_{n,n+1}], \quad n \geqslant 1. \tag{3.5}$$

$$E[Y_n] \geqslant E[Y_{m,n+m}], \quad m, n \geqslant 1. \tag{3.6}$$

Finally assume that for each $n \geqslant 1$ the sequence $X_{n,n+1} + Y_{n,n+1}, X_{n,n+2} + Y_{n,n+2}, \ldots$ is stochastically smaller than the sequence $X_1 + Y_1, X_2 + Y_2, \ldots$ Then

$$\lim_n \frac{1}{n} E[X_n] = \gamma \tag{3.7}$$

exists and is finite. Furthermore we have

$$E\left[\liminf_n \frac{1}{n}(X_n + Y_n)\right] \geqslant \gamma. \tag{3.8}$$

Proof. The proof of (3.7) is similar to that of (2.8). It remains to establish (3.8). We may assume that for each $k \geq 1$ there is a random variable U_k which is uniformly distributed on $\{1, 2, \ldots, k\}$ and has the property that

$$U_k \text{ is independent of } (X_{m,n}) \cup (Y_{m,n}), \quad k \geq 1. \tag{3.9}$$

Let us put (cf. [5, 6, 15] and [16, p. 279])

$$D_k^{(n)} = X_{k+U_n} - X_{k+U_n-1}, \quad k, n \geq 1.$$

By (3.9) and (3.7)

$$\lim_n E[D_k^{(n)}] = \gamma, \quad k \geq 1. \tag{3.10}$$

Using (3.1), (3.5), (3.6) and (3.10) we get

$$\sup_n E[|D_k^{(n)}|] < \infty, \quad k \geq 1. \tag{3.11}$$

Hence there exists a sequence $1 \leq n_1 < n_2 < \cdots$ such that $(D_1^{(n_i)}, D_2^{(n_i)}, \ldots)$ (conceived as a random element of R^∞) converges as $i \to \infty$ in distribution to some random element $(D_1, D_2, \ldots)$ of R^∞. Hence, by (3.9),

$$E[f(D_1, D_2, \ldots)] = \lim_i \frac{1}{n_i} \sum_{j=1}^{n_i} E[f(X_{j+1} - X_j, X_{j+2} - X_{j+1}, \ldots)], \quad f \in \mathcal{M}, \tag{3.12}$$

$\mathcal{M}$ denoting the family of all functions $f: R^\infty \to R$ which are continuous and bounded. Clearly (3.12) implies that(D_n) is stationary. Since, by (3.1) and the final hypothesis in Lemma 3.1,

$$\int_{D_1^{(n)} \geq a} D_1^{(n)} \, \mathrm{d}P \leq \int_{X_1 + Y_1 \geq a} (X_1 + Y_1) \, \mathrm{d}P, \quad n \geq 1, \; a > 0,$$

the $(D_1^{(n)})^+$ are uniformly integrable. Hence Fatou's lemma combined with (3.10) yields

$$\gamma \leq E[D_1] < \infty. \tag{3.13}$$

Applying Birkhoff's ergodic theorem to (D_n) we get that

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} D_j = D_\infty \quad \text{exists a.e. and in } L^1, \tag{3.14}$$

and, by (3.13),

$$E[D_\infty] \geq \gamma. \tag{3.15}$$

Combining (3.12) and (3.1) we get, taking into account the final hypothesis in Lemma 3.1, that, for all $f \in \mathcal{M}$ which are increasing,

$$E[f(D_1, D_1 + D_2, \ldots)] \leq E[f(X_1 + Y_1, X_2 + Y_2, \ldots)].$$

This implies (cf. the first remark in Section 2) that the sequence $D_1, D_1 + D_2, \ldots$ is stochastically smaller than $X_1 + Y_1, X_2 + Y_2, \ldots$ Hence (3.8) is a consequence of (3.14) and (3.15). This completes the proof of Lemma 3.1.

Let $p \geqslant 2$ be an integer. We will say that a sequence $(a_n) \subset R_+$ satisfies condition (Σ_p) provided

$$\sum_{n=1}^{\infty} \frac{1}{p^n m} a_{p^n m} < \infty \quad \text{for all } m \geqslant 1. \tag{3.16}$$

Note that if for a sequence $(a_n) \subset R_+$ we have that

$$a_n = \mathrm{O}\left(\frac{n}{\log n\,(\log\log n)^{1+\delta}}\right) \quad \text{as } n \to \infty$$

holds for some $\delta > 0$, (a_n) satisfies (Σ_p) for all $p \geqslant 2$. If $(E[Y_n])$ satisfies (Σ_p) for some $p \geqslant 2$, clearly

$$\lim_n \frac{1}{p^n m} Y_{p^n m} = 0 \text{ a.e.}, \quad m \geqslant 1.$$

Remark. If $p \geqslant 2$ and $q \geqslant 2$ are prime to one another, it is not difficult to construct a sequence $(a_n) \subset R_+$ satisfying (Σ_p) but not (Σ_q).

Our next result shows that under fairly general conditions certain subsequences of $((1/n)X_n)$ converge a.e. to the *same* random variable $\bar{X}_\infty$ given by (2.6) (note that we still do not require that (X_n) has any monotonicity properties).

Theorem 3.1. *Let the hypotheses of Theorem* 2.2 *and Lemma* 3.1 *be satisfied. Suppose further that* $(E[Y_n])$ *satisfies* (Σ_p) *for some* $p \geqslant 2$. *Then we have*

$$\lim_n \frac{1}{n} X_n = \bar{X}_\infty \quad \text{in } L^1 \tag{3.17}$$

and

$$\lim_n \frac{1}{p^n m} X_{p^n m} = \bar{X}_\infty \text{ a.e.}, \quad m \geqslant 1, \tag{3.18}$$

$\bar{X}_\infty$ *given by* (2.6).

Proof. The first assertion follows from Theorem 2.2. In order to prove (3.18), we first observe that by (2.17)

$$E\left[\limsup_n \frac{1}{p^n m} X_{p^n m}\right] \leqslant \gamma, \quad m \geqslant 1. \tag{3.19}$$

Noting that $(E[Y_n])$ satisfies (Σ_p), we deduce from (3.8)

$$E\left[\liminf_n \frac{1}{p^n m} X_{p^n m}\right] \geqslant \gamma, \quad m \geqslant 1.$$

Combining this with (3.19) and (2.16), we arrrive at (3.18).

We will show that (3.18) entails the almost everywhere convergence of the whole sequence $((1/n)X_n)$ provided (X_n) has certain monotonicity properties which we introduce now.

Definition. A sequence $(a_n) \subset R$ is called *quasi increasing* (*decreasing*) if there exists a constant $c \geqslant 0$ such that

$$a_k \leqslant a_{k+m} + cm, \quad k, m \geqslant 1, \tag{3.20}$$

or, respectively,

$$a_{k+m} \leqslant a_k + cm, \quad k, m \geqslant 1. \tag{3.21}$$

Note that (3.20) is equivalent to the requirement that the heights of the downward jumps $a_k - a_{k+1}$ are bounded above. An interesting example of a sequence of random variables (X_n) (arising in oriented percolation) which is quasi decreasing is given by (1.5) (compare Liggett [15] or Durrett [6] for further details).

Definition. A sequence $(a_n) \subset R$ is called *almost increasing* (*decreasing*) if there are sequences $(c_n) \subset R_+$ and $(d_n) \subset R$ both tending to zero as $n \to \infty$, such that

$$a_m \geqslant (1 + d_n) a_n \quad \text{if } m \geqslant (1 + c_n) n \tag{3.22}$$

or, respectively,

$$a_m \leqslant (1 + d_n) a_n \quad \text{if } m \geqslant (1 + c_n) n. \tag{3.23}$$

Example. If we put $a_n = (-1)^n$, $n \geqslant 1$, (a_n) is quasi increasing but not almost increasing. On the other hand, if we define a_n, $n \geqslant 1$, by

$$a_n = \begin{array}{ll} n - \sqrt{n} & \text{if } n \text{ is even,} \\ n & \text{otherwise,} \end{array}$$

(a_n) is almost increasing ((3.22) holds for $c_n \equiv 0$ and $d_n = -n^{-1/2}$) but not quasi increasing.

The proof of the following result is deferred to the next section.

Theorem 3.2. *Let (a_n) be a sequence which is quasi monotone or almost monotone. Suppose there exists a constant $-\infty \leqslant a \leqslant \infty$ and an integer $p \geqslant 2$ such that*

$$\lim_n \frac{1}{p^n m} a_{p^n m} = a \quad \textit{for all } m \geqslant 1.$$

Then the limit

$$\lim_n \frac{1}{n} a_n = a$$

exists.

When we say e.g. that a sequence of random variables is *pathwise quasi monotone* we mean that c in (3.10) (or (3.21)) as well as the kind of monotonicity may both depend on chance (an analogous remark applies to almost monotonicity). Theorems 3.1 and 3.2 can now be combined to yield our main convergence result (recall that $F_{m,n}$ and $G_{m,n}$ denote the distribution functions of $X^+_{m,n}$ and $Y_{m,n}$, respectively).

Theorem 3.3. *Suppose that (X_n) is either pathwise quasi monotone or pathwise almost monotone. Let $(E[Y_n])$ satisfy (Σ_p) for some $p \geq 2$ and let K be a positive constant. Assume that the following conditions hold:*

$$X_{n+m} \leq X_n + X_{n,n+m} + Y_{n,n+m}, \quad m, n \geq 1. \tag{3.24}$$

$$\lim_n \frac{1}{n} E[Y_n] = 0. \tag{3.25}$$

$$E[X_{jn,(j+1)n}] \geq -Kn, \quad j \geq 0, n \geq 1. \tag{3.26}$$

$$E[X_n] \geq E[X_{m,n+m}], \quad m, n \geq 1. \tag{3.27}$$

$$F_{0,n} \leq F_{m,n+m}, \quad m, n \geq 1. \tag{3.28}$$

$$G_{0,n} \leq G_{m,n+m}, \quad m, n \geq 1. \tag{3.29}$$

Finally assume that for each $j \geq 1$ the sequence $X_{j,j+1} + Y_{j,j+1}, X_{j,j+2} + Y_{j,j+2}, \ldots$ is stochastically smaller than $X_1 + Y_1, X_2 + Y_2, \ldots$ and that the sequences $X_{0,n}, X_{n,2n}, X_{2n,3n}, \ldots$ and $Y_{0,n}, Y_{n,2n}, Y_{2n,3n}, \ldots$ are both superstationary for each $n \geq 1$. Then $((1/n)X_n)$ converges a.e. and in L^1 to the random variable $\bar{X}_\infty$ given by (2.6).

Remark. In view of Lemmas (4.3) and (4.24) in Schürger [19] it is rather straightforward to deduce from Theorem 3.3 an a.e. limit theorem for a very general class of almost subadditive families of random sets [17, 19] which are 'pathwise quasi monotone' or 'pathwise almost monotone' in an obvious sense. This extends convergence results for random sets obtained e.g. in [2, 3, 19].

4. Proof of Theorem 3.2

In [21, p. 20] the convergence statement of Theorem 3.2 has been formulated in the case where $p = 2$ and (a_n) is monotone in the usual sense. Unfortunately, the proof given in [21] contains a gap since it is tacitly assumed there that (a_n) is nonnegative.

We will outline a proof of Theorem 3.2 only for quasi monotone sequences (the proof in the almost monotone case is longer but uses the same kind of arguments). Actually we will prove the following slightly stronger result.

Theorem 4.1. *Let (a_n) be a quasi monotone sequence. Suppose there exists a constant $a \in R$ and an integer $p \geqslant 2$ such that*

$$\limsup_n \frac{1}{p^n m} a_{p^n m} \leqslant a \quad \text{for all } m \geqslant 1. \tag{4.1}$$

Then we have

$$\limsup_n \frac{1}{n} a_n \leqslant a. \tag{4.2}$$

Proof. Let $p \geqslant 2$ be as in (4.1). We will make use of the simple fact that for all integers $m \geqslant 1$ and $n \geqslant pm$ there exist integers $t \geqslant 1$ and k such that

$$1 \leqslant k \leqslant (p-1)m \tag{4.3}$$

and

$$p^t(m+k-1) \leqslant n < p^t(m+k). \tag{4.4}$$

Let $0 < \varepsilon < 1$ be given. Fix any $m \geqslant 1$ such that

$$1 - \varepsilon < \frac{m}{m+1}, \qquad \frac{m+1}{m} < 1 + \varepsilon, \tag{4.5}$$

and let $n \geqslant pm$.

First assume that (a_n) is quasi increasing. By (3.20) and (4.4)

$$\frac{1}{n} a_n \leqslant \frac{a_{p^t(m+k)}}{p^t(m+k)} \frac{p^t(m+k)}{n} + c\left(\frac{p^t(m+k)}{n} - 1\right).$$

Since by (4.4), (4.3) and (4.5)

$$1 < \frac{p^t(m+k)}{n} < 1 + \varepsilon,$$

we get, by (4.2) and (4.3),

$$\limsup_n \frac{1}{n} a_n \leqslant \max(a+\varepsilon, (a+\varepsilon)(1+\varepsilon)) + c\varepsilon$$

and hence (4.2).

Now assume that (a_n) is quasi decreasing. By (3.21) and (4.4)

$$\frac{1}{n} a_n \leqslant \frac{a_{p^t(m+k-1)}}{p^t(m+k-1)} \frac{p^t(m+k-1)}{n} + c\left(1 - \frac{p^t(m+k-1)}{n}\right).$$

Since by (4.4), (4.3) and (4.5)

$$1 \geqslant \frac{p^t(m+k-1)}{n} > 1 - \varepsilon,$$

we get by (4.2) and (4.3)

$$\limsup_n \frac{1}{n} a_n \leqslant \max(a+\varepsilon, (a+\varepsilon)(1-\varepsilon)) + c\varepsilon$$

which again implies (4.2).

Acknowledgments

Part of this paper was written during the author's visit to the Department of Theoretical Mathematics of the Weizmann Institute of Science at Rehovot (Israel). I am indebted to Zvi Artstein for his hospitality. Finally, financial support from the Deutsche Forschungsgemeinschaft is gratefully acknowledged.

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